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Renormalization in Indefinite Metric Theories

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Abstract

We consider the Lee model, $V \leftrightarrows N + \theta$, and a two neutral scalar meson model with indefinite metrics. It is shown that in general the coupling will cause positive definite bare states to become negative length dressed states through renormalization effects. This can render the theory unphysical depending upon the auxiliary conditions imposed with the indefinite metric.

1. *Introduction*

Recently, interest has been stimulated in a general class of so-called indefinite metric field theories as possible remedies to certain divergence difficulties which still beset modern Quantum Electrodynamics and Weak Interaction Theory. Haller *et al.* (1967), have discussed this general question and most notably, Lee & Wick (1970) have proposed an entire theory of Quantum Electrodynamics free from the usual divergence difficulties. All such indefinite metric theories involve a subsidiary condition imposed upon the states making up the overall Hilbert space, and hence select only a subspace of such states as those which are physically realized in nature.

Aside from being full of mathematical complexities which require careful interpretation, such as the exponentially divergent transformation functionals in the recent Lee–Wick model (1969), negative metric theories can lead to completely unphysical results under certain circumstances. We shall consider the familiar Lee model with the metric specified beforehand as

$$\eta = \exp i\pi N^{\dagger} N \tag{1.1}$$

where $N^{\dagger}$ and N are the creation and annihilation operators of the N fermion ($V \leftrightarrows N + \theta$).

As is well known, the Lee model is only fully renormalizable when the bare coupling constant, g, is pure imaginary. Hence, the usual total Hamiltonian is not Hermitian, but as is easily seen, the total Hamiltonian is pseudo-Hermitian with respect to the metric η,

$$\eta H^{\dagger} \eta \equiv H^{*} = H \tag{1.2}$$

It shall be shown that in the presence of the interaction the physical V state becomes of negative length though the bare V particle was originally chosen to be of positive definite length. The interpretation is clear; in so choosing the interaction, we have dressed the bare V particle with a hoard of negative metric N particles to the extent that the physical V state has become 'extinguished'. We then consider a two complex scalar meson theory in which the usual charge conjugation operator emerges as the metric operator. This model exhibits clearly the mixing in of positive- and negative-length components into all particle states. The reader is referred to the excellent review of complex linear vector spaces with indefinite metric of Pandit (1959).

2. *Lee Model with an Indefinite Metric*

Following Lee we have the total Hamiltonian,

$$H = H_0 + H_1 \tag{2.1}$$

$$H_0 = m_V \int \psi_V^{\dagger} \psi_V \, d^3 r + m_N \int \psi_N^{\dagger} \psi_N \, d^3 r + (1/2) \int \{\pi^2 + \nabla\phi^2 + \mu^2 \phi^2\} \, d^3 r$$

$$H_1 = g \int \{\psi_V^{\dagger} \psi_N A(r) + \psi_N^{\dagger} \psi_V A^{\dagger}(r)\} \, d^3 r + \delta m_V \int \psi_V^{\dagger} \psi_V \, d^3 r$$

where $\psi_V^{\dagger}$, ψ_V and $\psi_N^{\dagger}$, ψ_N obey the usual anticommutation relations and their effects upon states are as follows,

$$\left.\begin{aligned} \psi_X^{\dagger}|\text{vac}\rangle &= \sqrt{(\delta^3)}\,(r)|\tilde{X}\rangle \\ \psi_X|\tilde{X}\rangle &= \sqrt{(\delta^3)}\,(r)|\text{vac}\rangle \end{aligned}\right\} \tag{2.2}$$

where ($\tilde{\ }$) denotes the bare particle state, and $|\text{vac}\rangle$ is the bare vacuum. $A(r)$ is given by

$$A(r) = \sum_k \sqrt{(2\omega R^{-1})}\, a_k \exp ik\,.\,r \tag{2.3}$$

where $a_k(a^{\dagger})$ destroys (creates) a θ particle of momentum k, and R is the volume of the system. m_V, m_N and μ are the observed masses of the respective particles. It is easily seen that, with g imaginary, we get the desired pseudo-Hermiticity condition, (1.2),

$$\eta H^{\dagger} \eta = H$$

It is easily seen from the foregoing that

$$H|N\rangle = H|\tilde{N}\rangle = m_N|N\rangle \tag{2.4}$$

where $|N\rangle$ is the physical ground state of the N particle. The metric causes N to have negative length, as

$$\langle N|\eta|N\rangle = -1 \tag{2.5}$$

as is easily verified.

We seek the physical V state, and write as usual,

$$|V\rangle = Z\left\{|\tilde{V}\rangle + g\sum_k f(k)a^\dagger|N\rangle\right\} \tag{2.6}$$

which leads straightforwardly to

$$\delta m_V = -|g|^2 \sum_k \sqrt{(2\omega R^{-1})}\{m_V - m_N - \omega\}^{-1} \tag{2.7}$$

$$f(k) = \sqrt{(2\omega R^{-1})}(m_V - m_N - \omega)^{-1} \tag{2.8}$$

by invoking the requirement that $|V\rangle$ be an eigenstate of the total Hamiltonian,

$$H|V\rangle = m_V|V\rangle \tag{2.9}$$

We now form the length of the particle V,

$$\langle V|\eta|V\rangle = |Z|^2\left\{\langle\tilde{V}|\tilde{V}\rangle + |g|^2\sum_k\sum_{k'} f^\dagger(k)f(k')\langle N|a_k\eta a_{k'}^\dagger|N\rangle\right\}$$

$$= |Z|^2\left\{1 - |g|^2\sum_k |f(k)|^2\right\} \tag{2.10}$$

and Z becomes

$$|Z| = \left|\left(1 - |g|^2\sum_k |f(k)|^2\right)\right|^{-1/2} \tag{2.11}$$

Hence, the overall sign of expression (2.10) is negative, owing to the presence of the negative-length N particles in the physical V state. We say that the V state has become 'extinguished' by the coupling to the unobservable negative-length N particle. If we carry this out to coupling constant renormalization, we find that the renormalized coupling, g_r, is given by

$$g_r^2 = |Z|^2|g|^2 \tag{2.12}$$

and is both finite and real.

3. *Two Scalar Meson Model*

We now investigate the admixture of positive- and negative-length components in a physical state in a theory with two scalar mesons, one of negative length bare state. Consider two complex scalar fields, ϕ', ψ, given by the following Fourier expansions,

$$\left.\begin{aligned} \phi &= \sqrt{(V^{-1})} \sum_p \sqrt{(2p_0^{-1})}\{b_p \exp ipx + \bar{b}_p^\dagger \exp -ipx\} \\ \psi &= \sqrt{(V^{-1})} \sum_k \sqrt{(2k_0^{-1})}\{a_k \exp ikx + \bar{a}_k^\dagger \exp -ikx\} \end{aligned}\right\} \tag{3.1}$$

where V is the volume of the system,

$$[a_k, a_{k'}^\dagger] = [\bar{a}_k, \bar{a}_{k'}^\dagger] = [b_k, b_{k'}^\dagger] = [\bar{b}_k, \bar{b}_{k'}^\dagger] = \delta_{kk'}$$

and all other commutators are zero. The total Hamiltonian density is

$$H(x) = H_1(x) + H_2(x) + H_3(x) \tag{3.2}$$

where

$$\left.\begin{aligned} H_1(x) &= \pi_1^\dagger \pi_1 + \nabla\phi^\dagger . \nabla\phi + \mu^2 \phi^\dagger \phi \\ H_2(x) &= \pi_2^\dagger \pi_2 + \nabla\psi^\dagger . \nabla\psi + m^2 \psi^\dagger \psi \\ H_3(x) &= g\{\phi(x)\psi^\dagger(x)\psi(x) - \phi^\dagger(x)\psi^\dagger(x)\psi(x)\} \end{aligned}\right\} \tag{3.3}$$

and m and μ are the bare masses. It is easily seen that the total Hamiltonian is not Hermitian, and the metric under which it becomes pseudo-Hermitian is seen to satisfy

$$\eta\phi^\dagger\eta = \phi, \qquad \eta\phi\eta = \phi^\dagger \tag{3.4}$$

or

$$\eta b_k^\dagger \eta = \bar{b}_{-k}, \qquad \eta \bar{b}_k^\dagger \eta = b_{-k}$$

which are seen to correspond to the usual definition of the charge conjugation operator, C, for the Φ field. Hence, the metric for this theory is C, and we have, as usual,

$$CH^\dagger C = H \tag{3.5}$$

where C operates only on the ϕ field. Hereon we use the Hamiltonians corresponding to (3.3) instead of the densities,

$$H_i = \int d^3 x H_i(x) \tag{3.6}$$

We may now construct projection operators for the positive- and negative-length components of state vectors. We form,

$$Q^\pm = (1/2)\{1 \pm C\} \tag{3.7}$$

whence, Q^+ (Q^-) annihilates negative- (positive-) length components in a given state.

4. *Physical States*

We consider the general (in) or (out) states of the ψ field which creates bare states of positive length. We have from the Lippmann–Schwinger equation (Schiff, 1955, p. 319),

$$|\psi\,\mathrm{in}\rangle = |\psi\,\mathrm{bare}\rangle + (E - H_0 + i\epsilon)^{-1} H_3' |\psi\,\mathrm{in}\rangle \tag{4.1}$$

and similarly for the (out) states, excepting for a minus sign in front of the $i\epsilon$ term above.

Now, we note from (3.7) and (3.3) that

$$Q^{\pm} H_3 = H_3 Q^{\mp} \tag{4.2}$$

and we define

$$\left.\begin{aligned} Q^{+}|\psi\,\mathrm{in}\rangle &= \alpha \\ Q^{-}|\psi\,\mathrm{in}\rangle &= \beta \end{aligned}\right\} \tag{4.3}$$

Also,

$$\left.\begin{aligned} Q^{+}|\psi\,\mathrm{bare}\rangle &= |\psi\,\mathrm{bare}\rangle \\ Q^{-}|\psi\,\mathrm{bare}\rangle &= 0 \end{aligned}\right\} \tag{4.4}$$

since the bare states of ψ have positive length. Hence, applying Q^{+} and Q^{-} to (4.1), and using (4.2)–(4.4), we derive,

$$\left.\begin{aligned} \alpha &= |\psi\,\mathrm{bare}\rangle + (E - H_0 + i\epsilon)^{-1} H_3 \beta \\ &\text{and} \\ \beta &= (E - H_0 + i\epsilon)^{-1} H_3 \alpha \end{aligned}\right\} \tag{4.5}$$

showing that an admixture of positive- and negative-length components necessarily results.

The condition that the physical (in) or (out) state be of positive length is given by,

$$\langle\alpha|\alpha\rangle - \langle\beta|\beta\rangle \geqslant 0 \tag{4.6}$$

If we now include a mass renormalization term, $M \equiv \delta m_\psi \int \psi^\dagger \psi d^3x$, within the interaction Hamiltonian, H_3, the bare ψ state in (4.1) becomes equivalent to the physical (in) or (out) state, and we obtain, as in (4.5),

$$\alpha = \alpha + (E - H_0 + i\epsilon)^{-1}\{H_3 \beta + M\alpha\}$$

or

$$\left.\begin{aligned} H_3 \beta + M\alpha &= 0 \\ &\text{and likewise,} \\ H_3 \alpha + M\beta &= 0 \end{aligned}\right\} \tag{4.7}$$

This result suggests that, in general, theories where the mass correction, M, is finite lead to

$$\left.\begin{aligned} \langle\alpha|H_3|\alpha\rangle = \langle\beta|H_3|\beta\rangle = 0 \\ \text{and,} \qquad \langle\beta|H_3|\alpha\rangle = \langle\alpha|H_3|\beta\rangle = -M \end{aligned}\right\} \tag{4.8}$$

that is, the mixing of the coupled state of negative length into the positive length bare state dominates over the level shift, which vanishes in first order by (4.8). Hence, in such a situation, we would expect the strength of the interaction to determine the overall sign on the length of the physical ψ states. In the case of M infinite, (4.8) no longer holds, though the same conditions may apply to the determination of the overall sign on the physical states.

Hence, in the general case of an indefinite metric theory where one imposes the auxiliary condition of regarding all negative-length states as unphysical (Sudarshan, 1961; Gupta, 1950), such as is commonplace a restriction in practice, one must beware that all states are not 'extinguished' by the renormalization effects. We have seen, further, that negative metric can be artificially introduced to supply a projection operator as in equation (3.7) for the study of renormalization effects upon physical states.

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